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BOX DIMENSION AND HAUSDORFF DIMENSION OF UNIFORM CANTOR SET

K. J. Shinde[1] & S. M. Padhye[2]

[1]*Research Scholar, Department of Mathematics, CAET, Dr. Panjabrao Deshmukh Krishi Vidyapeeth, Akola (MS), India*

[2]*Research Scholar, Department of Mathematics, RLT College of Science, Akola (MS), India*

ABSTRACT

The properties of uniform Cantor sets are proved and determined that Cantor ternary sets Cantor $-\frac{2}{5}$ *set, Cantor* $-\frac{3}{7}$ *set, Cantor* $-\frac{1}{4}$ *set, Cantor* $-\frac{1}{5}$ *set, Cantor* $-\frac{4}{15}$ *set, Cantor* $-\frac{1}{7}$ *set are uniform Cantor sets by using the definition of uniform Cantor set. Also, determined box dimension and Hausdorff dimension of uniform Cantor set as well as Cantor like sets.*

KEYWORDS: *Box Dimension, Cantor Set, Hausdorff Dimension*

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1. INTRODUCTION

Definition 1.1: Let F be any nonempty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The lower and upper box dimensions of F are defined respectively as $\underline{dim}_B(F) = \underline{lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta}$ and $\overline{dim}_B(F) = \overline{lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta}$.

Definition 1.2: Let m ≥ 2 be an integer and 0 $<\lambda<\frac{1}{m}$. Let I = [0 , 1]. We construct a Cantor like set by the following procedure. The set is then called as uniform Cantor set.

At first stage, from I we remove (m-1) intervals each of length $\frac{1-\lambda m}{m-1}$, leaving behind m equally spaced subintervals $I_{1,i}$ (1≤ i ≤ m) of lengths $\lambda|I| = \lambda$ i.e. $|I_{1,i}| = \lambda$, 1≤ i ≤ m .

The left end of $I_{1,1}$ coincides with left end of I and right end of $I_{1,m}$ coincides with right end of I. Let S_1 be union of subintervals $I_{1,i}$ (1≤ i ≤ m).

At second stage, we remove from each $I_{1,i}$ (1≤ i ≤ m), (m-1) intervals each of length $\frac{\lambda m(1-\lambda m)}{m(m-1)}$, leaving behind in all m^2 equally spaced subintervals $I_{2,1}$, $I_{2,2}$, ….. ,$I_{2,i}$ (1≤ i ≤ m^2) of equal lengths $\lambda|I_{1,i}|$ i.e. $|I_{2,i}| = \lambda^2$, 1≤ i ≤ m^2 . The extreme ends of subintervals coincide with the extreme ends of basic subintervals remaining at first stage. Let S_2 be union of subintervals $I_{2,1}$, $I_{2,2}$, ….. ,$I_{2,i}$ (1≤ i ≤ m^2).

At third stage, we remove from each $I_{2,i}$ (1≤ i ≤ m^2), (m-1) intervals each of length $\frac{\lambda^2 m^2(1-\lambda m)}{m^2(m-1)}$, leaving behind in all m^3 equally spaced subintervals $I_{3,1}$, $I_{3,2}$, ….. ,$I_{3,i}$ (1≤ i ≤ m^3) of equal lengths $\lambda|I_{2,i}|$ i.e. $|I_{3,i}| = \lambda^3$, 1≤ i ≤ m^3 .

The extreme ends of subintervals coincide with the extreme ends of basic subintervals remaining at second stage. Let S_3 be union of subintervals $I_{3,1}$, $I_{3,2}$, ….. , $I_{3,i}$ $(1\leq i \leq m^3)$.

Continuing in this way at k^{th} stage from each remaining interval, we remove (m-1) intervals each of length $\frac{\lambda^{k-1}m^{k-1}(1-\lambda m)}{m^{k-1}(m-1)}$, leaving behind in all m^k equally spaced subintervals $I_{k,1}$, $I_{k,2}$, ….. , $I_{k,i}$ $(1\leq i \leq m^k)$ of equal lengths $\lambda|I_{k-1,i}|$ i.e. $|I_{k,i}| = \lambda^k$, $1\leq i \leq m^k$. The extreme ends of subintervals coincide with the extreme ends of basic subintervals remaining at $(k-1)^{th}$ stage. Let S_k be union of subintervals $I_{k,1}$, $I_{k,2}$, ….. , $I_{k,i}$ $(1\leq i \leq m^k)$, $\therefore S_k = \bigcup_{i=1}^{m^k} I_{k,i}$, We define S = $\bigcap_{k=1}^{\infty} S_k$, Put s = $\frac{\log m}{-\log \lambda}$, Since $|I_{k+1,i}| = \lambda^{k+1}$

$$|I_{k+1,i}|^s = \lambda^{(k+1)s} = \lambda^{ks}\lambda^s \quad \text{…………………………………………………………………………………..……… (i)}$$

But s = $\frac{\log m}{-\log \lambda}$, $\quad$ s $\log\lambda = -\log m$, $\quad$ $\log\lambda^s = -\log m^{-1}$, $\lambda^s = m^{-1}$

Putting in equation (i) we get, $|I_{k+1,i}|^s = \lambda^{ks}m^{-1}$

$$|I_{k+1,i}|^s = \frac{\lambda^{ks}}{m} \quad \text{…………………………………………………………………………………………….... (ii)}$$

But $|I_{k,i}| = \lambda^k$

$$|I_{k,i}|^s = \lambda^{ks}$$

Putting in equation (ii) we get,

$$|I_{k+1,i}|^s = \frac{1}{m}|I_{k,i}|^s, \ 1\leq i \leq m^k$$

Theorem 1.3: Let $\{E_i\}$ be a sequence of measurable sets. We have

- If $E_1 \subseteq E_2 \subseteq$ ….. , then $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim \mu(E_n)$.
- If $E_1 \supseteq E_2 \supseteq$ ….. , then $\mu(E_i)<\infty$, $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim \mu(E_n)$.

2. MEASURE OF UNIFORM CANTOR SET

Now, we determine the measure of uniform Cantor set in two ways. In the first way, we use the length of remaining closed intervals at each stage,andin second way,we use the length of removed open intervals at each stage.

Theorem 2.1: If S is uniform Cantor set, then m(S) = 0.

Proof: At first stage, remaining m equally spaced subintervals are each of length λ.

∴ The sum of the lengths of remaining closed intervals at first stage = $m(S_1)$ = mλ.

At second stage, remaining m^2 equally spaced subintervals are each of length λ^2.

∴ The sum of the lengths of remaining closed intervals at second stage = $m(S_2) = m^2\lambda^2$.

At third stage, remaining m^3 equally spaced subintervals are each of length λ^3.

∴ The sum of the lengths of remaining closed intervals at third stage = $m(S_3) = m^3\lambda^3$.

Continuing in this way, at k^{th} stage remaining m^k equally spaced subintervals are each of length λ^k .

∴ The sum of the length of remaining closed intervals at k^{th} stage = $m(S_k) = m^k\lambda^k$.

Since S = $\bigcap S_k$ and using Theorem 1.3 we get, m(S) = m($\bigcap S_k$)= $\lim_{k\to\infty} m(S_k)$

= $\lim_{k\to\infty} m^k\lambda^k$ ($\because \lambda<\frac{1}{m} \Rightarrow m\lambda < 1 \Rightarrow m^k\lambda^k < 1$) ∴ m(S) = 0

∴ Lebesgue measure of uniform Cantor set is zero.

Alternative proof: At first stage, we remove m-1 open intervals each of length $\frac{1-\lambda m}{m-1}$.

∴ The sum of the lengths of the removed intervals at first stage = (m-1)($\frac{1-\lambda m}{m-1}$) = $1-\lambda m$.

At second stage, we remove m(m-1) open intervals each of length $\frac{\lambda m(1-\lambda m)}{m(m-1)}$.

∴ The sum of the lengths of the removed intervals at second stage = m(m-1) $\frac{\lambda m(1-\lambda m)}{m(m-1)}$ = $\lambda m(1-\lambda m)$.

At third stage, we remove m^2(m-1) open intervals each of length $\frac{\lambda^2 m^2(1-\lambda m)}{m^2(m-1)}$.

∴ The sum of the lengths of the removed intervals at third stage = m^2(m-1) $\frac{\lambda^2 m^2(1-\lambda m)}{m^2(m-1)}$ = $\lambda^2 m^2(1-\lambda m)$ and so on.

∴ The sum of the lengths of the removed intervals in the geometric construction of uniform Cantor set S

= $(1-\lambda m)$ + $\lambda m(1-\lambda m)$+ $\lambda^2 m^2(1-\lambda m)$ + ……

= $(1-\lambda m)$ [1 + λm + $\lambda^2 m^2$ + ……] =$(1-\lambda m)$ $\frac{1}{1-\lambda m}$ = 1

Therefore, m(S) = m(I) – m(All removed open intervals)

= 1 – 1 = 0 (since m(I) = 0)

Therefore m(S) = 0

Hence, Lebesgue measure of uniform Cantor set is zero.

Theorem 2.2: Cantor ternary set, Cantor – $\frac{2}{5}$ set, Cantor – $\frac{3}{7}$ set, Cantor – $\frac{1}{4}$ set, Cantor – $\frac{1}{5}$ set, Cantor – $\frac{4}{15}$ set, Cantor – $\frac{1}{7}$ set are Uniform Cantor Set.

Proof: We show that Cantor ternary set, Cantor - $\frac{2}{5}$ set, Cantor - $\frac{3}{7}$ set, Cantor – $\frac{1}{4}$ set, Cantor - $\frac{1}{5}$ set Cantor – $\frac{1}{4}$ set are uniform Cantor sets. We verify that $|I_{k+1,i}|^s = \frac{1}{m}|I_{k,i}|^s$

i) For Cantor ternary set

Here, m = 2, $\lambda = \frac{1}{3}$ and s = $\frac{\log 2}{\log 3}$, Here $|I_{k,i}| = (\frac{1}{3})^k$

Therefore, $|I_{k,i}|^s = (\frac{1}{3})^{ks}$ ……………………………………………………………………..………. (1)

Now, $|I_{k+1,i}|^s = (\frac{1}{3})^{(k+1)s} = (\frac{1}{3})^{ks}(\frac{1}{3})^s$…………………………………………………………………….………. (2)

But s = $\frac{\log 2}{\log 3}$, $\therefore$ s log 3 = log 2, $\log 3^s = \log 2$, $3^s = 2$

Putting in equation (2) we get $|I_{k+1,i}|^s = \frac{1}{2}\frac{1}{3^{ks}}$ i.e. $|I_{k+1,i}|^s = \frac{1}{m}|I_{k,i}|^s$

$\therefore$ Cantor ternary set is uniform Cantor set.

ii) For Cantor – $\frac{2}{5}$ set

Here, m = 2, $\lambda = \frac{2}{5}$ and s = $\frac{\log 2}{-\log(\frac{2}{5})}$, Here $|I_{k,i}| = (\frac{2}{5})^k$

$|I_{k,i}|^s = (\frac{2}{5})^{ks}$ …………………………………………………………………… (3)

Now, $|I_{k+1,i}|^s = (\frac{2}{5})^{(k+1)s} = (\frac{2}{5})^{ks}(\frac{2}{5})^s$ …………………………………………………… (4)

But s = $\frac{\log 2}{-\log(\frac{2}{5})}$, $\therefore$ s $\log(\frac{2}{5}) = -\log 2$, $\log(\frac{2}{5})^s = \log 2^{-1}$, $(\frac{2}{5})^s = \frac{1}{2}$

Putting in equation (4) we get,

$|I_{k+1,i}|^s = \frac{1}{2}(\frac{2}{5})^{ks}$ i.e. $|I_{k+1,i}|^s = \frac{1}{m}|I_{k,i}|^s$

$\therefore$ Cantor - $\frac{2}{5}$ set is uniform Cantor set.

iii) For Cantor – $\frac{3}{7}$ set

Here, m = 2, $\lambda = \frac{3}{7}$ and s = $\frac{\log 2}{-\log(\frac{3}{7})}$, Here $|I_{k,i}| = (\frac{3}{7})^k$

$|I_{k,i}|^s = (\frac{3}{7})^{ks}$ …………………………………………………………………… (5)

Now $|I_{k+1,i}|^s = (\frac{3}{7})^{(k+1)s} = (\frac{3}{7})^{ks}(\frac{3}{7})^s$ …………………………………………………… (6)

But s = $\frac{\log 2}{-\log(\frac{3}{7})}$, $\therefore$ s $\log(\frac{3}{7}) = -\log 2$, $\log(\frac{3}{7})^s = \log 2^{-1}$, $(\frac{3}{7})^s = \frac{1}{2}$

Putting in equation (6) we get

$|I_{k+1,i}|^s = \frac{1}{2}(\frac{3}{7})^{ks}$ i.e. $|I_{k+1,i}|^s = \frac{1}{m}|I_{k,i}|^s$

$\therefore$ Cantor - $\frac{3}{7}$ set is uniform Cantor set.

iv) For Cantor – $\frac{1}{4}$ set

Here, m = 2, $\lambda = \frac{1}{4}$ and s = $\frac{1}{2}$, Here $|I_{k,i}| = (\frac{1}{4})^k$

$|I_{k,i}|^s = 4^{-ks}$ …………………………………………………………………… (7)

Now, $|I_{k+1,i}|^s = 4^{-(k+1)s} = 4^{-ks}4^{-s}$ …………………………………………………… (8)

But s = $\frac{1}{2}$

Putting in equation (8) we get $|I_{k+1,i}|^s = 4^{-ks}4^{-\frac{1}{2}} = \frac{1}{2}4^{-ks}$ i.e. $|I_{k+1,i}|^s = \frac{1}{m}|I_{k,i}|^s$

∴ Cantor - $\frac{1}{4}$ set is uniform Cantor set.

v) For Cantor – $\frac{1}{5}$ set

Here, m = 3, $\lambda = \frac{1}{5}$ and s = $\frac{\log 3}{\log 5}$, Here $|I_{k,i}| = (\frac{1}{5})^k$

$|I_{k,i}|^s = (\frac{1}{5})^{ks}$ ………………………………………………………………………………………… (9)

Now $|I_{k+1,i}|^s = (\frac{1}{5})^{(k+1)s} = (\frac{1}{5})^{ks}(\frac{1}{5})^s$ ……………………………………………………………………… (10)

But s = $\frac{\log 3}{\log 5}$, ∴ $s \log 5 = \log 3$, $\log 5^s = \log 3$, $5^s = 3$

Putting in equation (10) we get $|I_{k+1,i}|^s = \frac{1}{3}\frac{1}{5^{ks}}$ i.e. $|I_{k+1,i}|^s = \frac{1}{m}|I_{k,i}|^s$

∴ Cantor - $\frac{1}{5}$ set is uniform Cantor set.

vi) For Cantor – $\frac{4}{15}$ set

Here, m = 3, $\lambda = \frac{4}{15}$ and s = $\frac{-\log 3}{\log(\frac{4}{15})}$

Here, $|I_{k,i}| = (\frac{4}{15})^k$

$|I_{k,i}|^s = (\frac{4}{15})^{ks}$ ………………………………………………………………………………………… (11)

Now, $|I_{k+1,i}|^s = (\frac{4}{15})^{(k+1)s} = (\frac{4}{15})^{ks}(\frac{4}{15})^s$ …………………………………………………………………… (12)

But s = $\frac{-\log 3}{\log(\frac{4}{15})}$, ∴ $s \log(\frac{4}{15}) = -\log 3$, $\log(\frac{4}{15})^s = \log 3^{-1}$, $(\frac{4}{15})^s = 3^{-1}$

Putting in equation (12) we get

$|I_{k+1,i}|^s = \frac{1}{3}(\frac{4}{15})^{ks}$ i.e. $|I_{k+1,i}|^s = \frac{1}{m}|I_{k,i}|^s$

∴ Cantor - $\frac{4}{15}$ set is uniform Cantor set.

vii) For Cantor – $\frac{1}{7}$ set

Here, m = 4, $\lambda = \frac{1}{7}$ and s = $\frac{\log 4}{\log 7}$

Here, $|I_{k,i}| = (\frac{1}{7})^k$

$|I_{k,i}|^s = (\frac{1}{7})^{ks}$ ………………………………………………………………………………………… (13)

Now, $|I_{k+1,i}|^s = (\frac{1}{7})^{(k+1)s} = (\frac{1}{7})^{ks}(\frac{1}{7})^s$ ……………………………………………………………………… (14)

But s = $\frac{\log 4}{\log 7}$, ∴ s $\log 7 = \log 4$, $\log 7^s = \log 4$, $7^s = 4$

Putting in equation (14) we get

$|I_{k+1,i}|^s = \frac{1}{4}\frac{1}{7^{ks}}$ i.e. $|I_{k+1,i}|^s = \frac{1}{m}|I_{k,i}|^s$

∴ Cantor - $\frac{1}{7}$ set is uniform Cantor set.

3. BOX DIMENSION AND HAUSDORFF DIMENSION OF UNIFORM CANTOR SET

Proposition 3.1: Let s be a number strictly between 0 and 1 and F be a uniform Cantor set defined in definition of 1.2.Then,$dim_H(F)$ = s and 0 <$H^s(F)$<∞.

Proposition 3.2: If F is uniform Cantor set then $dim_H(F) = dim_B(F) = \frac{\log m}{-\log \lambda}$.

Proof: Let F be uniform Cantor set, 0 <λ<$\frac{1}{m}$ and s = $\frac{\log m}{-\log \lambda}$.∴ λ<$\frac{1}{m}$, m <$\frac{1}{\lambda}$, $\log m$<$-\log \lambda$

∴ $\frac{\log m}{-\log \lambda}$< 1 (∵ λ<$\frac{1}{m}$ ⟹ $\log \lambda$ is negative and $-\log \lambda$ is positive)

∴ s < 1 ……………………………………………………………………………………………….(1)

Now 0 <λ, $\log 0$<$\log \lambda$,∞<$\log \lambda$, $\log \lambda$< 0

∴ $-\log \lambda$> 0 and m > 2 ⟹ $\log m$> 0

∴ $\frac{\log m}{-\log \lambda}$> 0

∴ s > 0 ……………………………………………………………………………………………(2)

From equations (1) and (2) we get0 <$\frac{\log m}{-\log \lambda}$< 1 i.e. 0 < s < 1

By proposition 3.1 we get,

$dim_H(F)$ = s ……………..……………………………………………………………………… (3)

And, 0 <$H^s(F)$<∞

Now, we show that $dim_B(F)$ = s. We first show that $\overline{dim}_B(F)$ ≤ s.

By definition 1.1 we get,

$\overline{dim}_B(F) = \overline{lim}_{\delta_k \to 0} \frac{\log N_{\delta_k}(F)}{-\log \delta_k}$.

We know that F is covered by m^k basic intervals of length λ^k inE_k for each k.

∴ $N_{\delta_k}(F) \leq m^k$ and $\delta_k = \lambda^k$

$$\therefore \overline{dim}_B(F) \leq \overline{lim}_{k\to\infty} \frac{\log m^k}{-\log \lambda^k}$$

$= \overline{lim}_{k\to\infty} \frac{k \log m}{-k\log \lambda}$

$\overline{dim}_B(F) \le \frac{\log m}{-\log\lambda}$ = s ……………………………………………………..……..(4)

We get, $dim_H(F) \le \underline{dim}_B(F) \le \overline{dim}_B(F)$

From equations (3) and (4) we get, s $\le \underline{dim}_B(F) \le \overline{dim}_B(F) \le$ s

$\therefore dim_B(F)$ = s

$\therefore dim_H(F) = dim_B(F)$ = s

We consider now, several particular cases for different values of m and λ to determined Hausdorff dimension and box dimension of different Cantor like sets by using Proposition 3.2.

Particular Case 3.3 When m = 2, $\lambda = \frac{1}{3}$

By Proposition 3.2 we get,

$dim_H(F) = dim_B(F) = \frac{\log m}{-\log\lambda} = \frac{\log 2}{-\log(\frac{1}{3})} = \frac{\log 2}{\log 3}$

which is Hausdorff dimension and box dimension of Cantor ternary set.

Particular Case 3.4When m = 2, $\lambda = \frac{2}{5}$

By Proposition 3.2 we get,

$dim_H(F) = dim_B(F) = \frac{\log m}{-\log\lambda} = \frac{\log 2}{-\log(\frac{2}{5})} = \frac{\log 2}{\log(\frac{5}{2})}$

which is Hausdorff dimension and box dimension of Cantor - $\frac{2}{5}$ set

Particular Case 3.4 When m = 2, $\lambda = \frac{3}{7}$

By Proposition 3.2 we get,

$dim_H(F) = dim_B(F) = \frac{\log m}{-\log\lambda} = \frac{\log 2}{-\log(\frac{3}{7})} = \frac{\log 2}{\log(\frac{7}{3})}$

which is Hausdorff dimension and box dimension of Cantor - $\frac{3}{7}$ set.

Particular Case 3.5 When m = 2, $\lambda = \frac{1}{4}$

By Proposition 3.2 we get,

$dim_H(F) = dim_B(F) = \frac{\log m}{-\log\lambda} = \frac{\log 2}{-\log(\frac{1}{4})} = \frac{1}{2}$

which is Hausdorff dimension and box dimension of Cantor - $\frac{1}{4}$ set.

Particular Case 3.6 When m = 3, $\lambda = \frac{1}{5}$

By Proposition 3.2 we get,

$$dim_H(F) = dim_B(F) = \frac{\log m}{-\log \lambda} = \frac{\log 3}{-\log(\frac{1}{5})} = \frac{\log 3}{\log 5}$$

which is Hausdorff dimension and box dimension of Cantor - $\frac{1}{5}$ set.

Particular Case 3.7 When m = 3, $\lambda = \frac{4}{15}$

By Proposition 3.2 we get,

$$dim_H(F) = dim_B(F) = \frac{\log m}{-\log \lambda} = \frac{\log 3}{-\log(\frac{4}{15})} = \frac{\log 3}{\log(\frac{15}{4})}$$

which is Hausdorff dimension and box dimension of Cantor - $\frac{4}{15}$ set.

Particular Case 3.8When m = 4, $\lambda = \frac{1}{7}$

By Proposition 3.2 we get,

$$dim_H(F) = dim_B(F) = \frac{\log m}{-\log \lambda} = \frac{\log 4}{-\log(\frac{1}{7})} = \frac{\log 4}{\log 7}$$

which is Hausdorff dimension and box dimension of Cantor - $\frac{1}{7}$ set

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